The Density Ratio of Poisson Binomial versus Poisson Distributions

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Abstract

Let $b(x)$ be the probability that a sum of independent Bernoulli random variables with parameters $p_1, p_2, p_3, \ldots \in [0,1)$ equals x, where $\lambda := p_1 + p_2 + p_3 + \cdots$ is finite. We prove two inequalities for the maximum of the density ratio $b(x)/\pi_\lambda(x)$, where π_λ is the probability mass function of the Poisson distribution with parameter λ.

Key words: Poisson approximation, relative errors, total variation distance.

1 Introduction and main results

We consider independent Bernoulli random variables $Z_1, Z_2, Z_3, \ldots \in \{0,1\}$ with parameters $\mathbb{P}(Z_i = 1) = \mathbb{E}(Z_i) = p_i \in [0,1)$ and their sum $X = \sum_{i\ge 1} Z_i$. By the first and second Borel–Cantelli lemmas, X is almost surely finite if and only if the sequence $\boldsymbol{p} = (p_i)_{i\ge 1}$ satisfies

$$\lambda := \sum_{k=1}^{\infty} p_k \ < \ \infty, \tag{1}$$

and we exclude the trivial case $\lambda = 0$. Under this assumption, the distribution $Q = Q_{\boldsymbol{p}}$ of X is given by

$$b(x) = b_{\boldsymbol{p}}(x) := \mathbb{P}(X = x) \ = \ \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} p_i \prod_{k \in J^c} (1 - p_k) \tag{2}$$

for integers $x \ge 0$, where $\mathcal{J}(x) := \{J \subset \mathbb{N} : \#J = x\}$ and $J^c := \mathbb{N} \setminus J$.

It is well-known that the distribution Q may be approximated by the Poisson distribution Poiss_λ with probability mass function $\pi = \pi_\lambda$ given by $\pi(x) = e^{-\lambda}\lambda^x/x!$, provided

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that the quantity

$$\Delta \ := \ \lambda^{-1} \sum_{i \ge 1} p_i^2$$

is small. Indeed, Barbour and Hall (1984) obtained the remarkable bound

$$d_{\mathrm{TV}}(Q, \mathrm{Poiss}_\lambda) \ \le \ (1 - e^{-\lambda}) \Delta$$

via a suitable version of Stein's method developed by Chen (1975). Here $d_{\mathrm{TV}}(\cdot, \cdot)$ stands for total variation distance. Note also that $\mathrm{Var}(X) = \sum_{i \ge 1} p_i(1 - p_i) = \lambda(1 - \Delta)$, and

$$\Delta \ \le \ p_* \ := \ \max_{i \ge 1} p_i.$$

Main results. Motivated by Dümbgen et al. (2020), we are aiming at upper bounds for the maximal density ratio

$$\rho(Q, \mathrm{Poiss}_\lambda) \ := \ \sup_{x \ge 0} r(x)$$

with $r(x) = r_{\boldsymbol{p}}(x) := b(x)/\pi(x)$. Note that the probability mass functions b and π are densities (in the sense of the Radon-Nikodym theorem) of Q and Poiss_λ with respect to counting measure on the set $\mathbb{N}_0$ of nonnegative integers. Thus $r = b/\pi_\lambda$ is the "density ratio" in the title. For arbitrary sets $A \subset \mathbb{N}_0$, the probability $Q(A) = \mathbb{P}(X \in A)$ is never larger than the corresponding Poisson probability times $\rho\bigl(Q, \mathrm{Poiss}_\lambda\bigr)$, no matter how small the Poisson probability is. Hence, $\rho(Q, \mathrm{Poiss}_\lambda)$ is a strong measure of error when Q is approximated by Poiss_λ, see also Remark 3 below. While Dümbgen et al. (2020) obtained explicit and essentially sharp bounds for $\rho(Q, P)$ for various pairs of distributions P and Q, the present setting with the particular Poisson binomial distribution Q and $P = \mathrm{Poiss}_\lambda$ seems to be substantially more difficult. In this note we prove the following result:

Theorem 1. *For any sequence $\boldsymbol{p}$ of probabilities $p_i \in [0, 1)$ with $\lambda = \sum_{i \ge 1} p_i < \infty$,*

$$\rho(Q, \mathrm{Poiss}_\lambda) \ \le \ (1 - p_*)^{-1}.$$

We conjecture that Theorem 1 is true with Δ in place of p_*. In the case of $\lambda \le 1$ we can prove the following result:

Theorem 2. *For any sequence $\boldsymbol{p}$ of probabilities $p_i \in [0, 1)$ with $\lambda = \sum_{i \ge 1} p_i \le 1$,*

$$\Delta\Bigl(1 - \frac{\Delta}{2} - \frac{\lambda}{2(1 - p_*)}\Bigr) \ \le \ \log \rho(Q, \mathrm{Poiss}_\lambda) \ \le \ \Delta.$$

In particular, $\lambda \le 1$ implies that $\rho(Q, \mathrm{Poiss}_\lambda) \le e^\Delta < 1/(1 - \Delta)$. And since $\Delta \le p_* \le \lambda$, Theorem 2 implies that

$$\frac{\log \rho(Q, \mathrm{Poiss}_\lambda)}{\Delta} \ \to \ 1 \quad \text{as } \lambda \to 0.$$

Remark 3 (Total variation distance). Proposition 1 (a) of Dümbgen et al. (2020) implies that $d_{\mathrm{TV}}(Q, \mathrm{Poiss}_\lambda) \le Q(\{b > \pi\})\bigl(1 - \rho(Q, \mathrm{Poiss}_\lambda)^{-1}\bigr)$. Since $b(0) = \prod_{i\ge 1}(1 - p_i)$ satisfies the two inequalities $1 - \lambda \le b(0) < e^{-\lambda} = \pi(0)$, we obtain the inequality $Q(\{b > \pi\}) \le 1 - b(0) \le \min(1, \lambda)$ and the bounds

$$\begin{aligned} d_{\mathrm{TV}}(Q, \mathrm{Poiss}_\lambda) \ &\le \ \min(1,\lambda)\bigl(1 - \rho(Q, \mathrm{Poiss}_\lambda)^{-1}\bigr) \\ &\le \ \begin{cases} \min(1,\lambda) p_* \\ \lambda(1 - e^{-\Delta}) \ \le \ \lambda\Delta = \sum_{i\ge 1} p_i^2 \quad \text{if } \lambda \le 1. \end{cases} \end{aligned}$$

The remainder of this note is structured as follows: In Section 2 we provide some basic formulae for the probability masses $b(x)$ and the ratios $r(x)$. Then we present the proofs of Theorems 1 and 2 in Section 3.

2 Auxiliary results

2.1 The probability mass function of Q

Since $b(0) < 1$ (see Remark 3), we know that $\rho(Q, \mathrm{Poiss}_\lambda) = \sup_{x\ge 1} r(x)$. Writing

$$\prod_{i\in J} p_i \prod_{k\in J^c} (1 - p_k) \ = \ \prod_{i\in J} \frac{p_i}{1 - p_i} \prod_{k\ge 1} (1 - p_k) \ = \ b(0) \prod_{i\in J} \frac{p_i}{1 - p_i},$$

equation (2) may be reformulated as

$$b(x) \ = \ b(0) \sum_{J\in\mathcal{J}(x)} W(J)$$

with

$$W(J) \ := \ \prod_{i\in J} q_i \quad \text{and} \quad q_i \ := \ \frac{p_i}{1 - p_i} \ \in \ [0, \infty),$$

i.e. $p_i = q_i/(1 + q_i)$. Note also that the support of Q is equal to an integer interval containing 0. Precisely,

$$b(x) \ > \ 0 \quad \text{if and only if} \quad x \ \le \ \#\{i \ge 1 : p_i > 0\} \in \mathbb{N} \cup \{\infty\}.$$

2.2 Discrete scores

For any $x \ge 0$,

$$\frac{\pi(x+1)}{\pi(x)} \ = \ \frac{\lambda}{x+1},$$

so the “scores” $r(x+1)/r(x)$ are given by

$$\frac{r(x+1)}{r(x)} \ = \ \frac{(x+1)b(x+1)}{\lambda b(x)}$$

for $x \geq 0$ with $b(x) > 0$. If x_o is a maximizer of $r(\cdot)$, then

$$\frac{(x_o+1)b(x_o+1)}{b(x_o)} \ \leq \ \lambda \ \leq \ \frac{x_o b(x_o)}{b(x_o-1)} \tag{3}$$

with $b(-1) := 0$.

There are various ways to represent the ratios $b(x+1)/b(x)$. The following notation will be useful for that task: For any set $J \subset \mathbb{N}$, we define

$$s(J) \ := \ \sum_{i \in J} p_i \quad \text{and} \quad S(J) \ := \ \sum_{i \in J} q_i.$$

In case of $x := \#J < \infty$ we set

$$\bar{s}(J) \ := \ s(J)/x, \quad \bar{S}(J) \ := \ S(J)/x \quad \text{and} \quad \bar{W}(J) \ := \ W(J) \Big/ \sum_{L \in \mathcal{J}(x)} W(L)$$

with the convention $0/0 := 0$. The numbers $\bar{W}(J)$ are probability weights in the sense that $\sum_{J \in \mathcal{J}(x)} \bar{W}(J) = 1$ whenever $b(x) > 0$. In that case,

$$\begin{aligned}
\frac{b(x+1)}{b(0)} = \sum_{L \in \mathcal{J}(x+1)} W(L) \ &= \ \sum_{L \in \mathcal{J}(x+1)} \frac{1}{x+1} \sum_{k \in L} W(L \setminus \{k\}) q_k \\
&= \ \frac{1}{x+1} \sum_{J \in \mathcal{J}(x)} W(J) \sum_{k \in J^c} q_k \\
&= \ \frac{1}{x+1} \sum_{J \in \mathcal{J}(x)} W(J) S(J^c).
\end{aligned}$$

Consequently,

$$\frac{(x+1)b(x+1)}{b(x)} \ = \ \sum_{J \in \mathcal{J}(x)} \bar{W}(J) S(J^c). \tag{4}$$

Alternatively, if $b(x+1) > 0$, then

$$\begin{aligned}
\frac{b(x)}{b(0)} = \sum_{J \in \mathcal{J}(x)} W(J) \ &= \ \sum_{J \in \mathcal{J}(x)} W(J) \sum_{k \in J^c} \frac{q_k}{S(J^c)} \\
&= \ \sum_{J \in \mathcal{J}(x)} \sum_{k \in J^c} \frac{W(J \cup \{k\})}{q_k + S((J \cup \{k\})^c)} \\
&= \ \sum_{L \in \mathcal{J}(x+1)} W(L) \sum_{k \in L} \frac{1}{q_k + S(L^c)}.
\end{aligned}$$

Consequently,

$$\frac{b(x)}{(x+1)b(x+1)} \ = \ \sum_{L \in \mathcal{J}(x+1)} \bar{W}(L) \frac{1}{x+1} \sum_{k \in L} \frac{1}{q_k + S(L^c)}. \tag{5}$$

One can repeat the previous arguments with the sums $\sum_{k\in J^c} p_j/s(J^c) = 1$ in place of $\sum_{k\in J^c} q_k/S(J^c) = 1$. This leads to

$$\frac{b(x)}{b(0)} \;=\; \sum_{J\in\mathcal{J}(x)}\sum_{k\in J^c}\frac{W(J)p_k}{p_k+s((J\cup\{k\})^c)} \;=\; \sum_{L\in\mathcal{J}(x+1)} W(L)\sum_{k\in L}\frac{1-p_k}{p_k+s(L^c)},$$

because $W(J)p_k = W(J\cup\{k\})(1-p_k)$ for $k\in J^c$. Consequently,

$$\frac{b(x)}{(x+1)b(x+1)} \;=\; \sum_{L\in\mathcal{J}(x+1)} \bar{W}(L)\frac{1}{x+1}\sum_{k\in L}\frac{1-p_k}{p_k+s(L^c)}. \tag{6}$$

Analyzing equation (6) leads to a first result about the location of maximizers of $r(\cdot)$:

Proposition 1. *Any maximizer $x_o\in\mathbb{N}_0$ of $r(\cdot)$ satisfies the inequalities $1\le x_o\le\lceil\lambda\rceil$.*

Proof of Proposition 1. The inequality $x_o\ge 1$ follows from $r(0)<1$, see Remark 3. To verify the inequality $x_o\le\lceil\lambda\rceil$, it suffices to show that $r(x+1)/r(x)<1$ for any integer $x\ge\lambda$ with $b(x)>0$. This is equivalent to

$$\frac{b(x)}{(x+1)b(x+1)} \;>\; \lambda^{-1}. \tag{7}$$

If $b(x+1)=0$, this inequality is trivial. Otherwise, the left hand side of (7) is given by (6). Since $(1-y)/(y+s(L^c))$ is a strictly convex function of $y\ge 0$, Jensen's inequality implies that

$$\frac{1}{x+1}\sum_{k\in L}\frac{1-p_k}{p_k+s(L^c)} \;>\; \frac{1-\bar{s}(L)}{\bar{s}(L)+s(L^c)} \;=\; \frac{1-\bar{s}(L)}{\bar{s}(L)+\lambda-s(L)} \;=\; \frac{1-\bar{s}(L)}{\lambda-x\bar{s}(L)}.$$

But in case of $x\ge\lambda$,

$$\frac{1-\bar{s}(L)}{\lambda-x\bar{s}(L)} \;\ge\; \frac{1-\bar{s}(L)}{\lambda-\lambda\bar{s}(L)} \;=\; \lambda^{-1},$$

whence (7) holds true. $\square$

Finally, let us mention that the probability mass function b is ultra-log-concave in the sense that $\log r = \log(b/\pi)$ is concave, i.e. $r(x+1)/r(x)$ is monotone decreasing in $x\in\{y\ge 0: b(y)>0\}$, see Section 4 of Saumard and Wellner (2014) and the references therein. Equivalently, $(x+1)b(x+1)/b(x)$ is monotone decreasing in $x\in\{y\ge 0: b(y)>0\}$. With a direct argument one can even show a stronger result.

Proposition 2. *The ratio $(x+1)b(x+1)/b(x)$ is strictly decreasing in $x\in\{y\ge 0: b(y)>0\}$.*

Proof of Proposition 2. We have to show that for any integer $x\ge 0$ with $b(x+1)>0$,

$$\frac{(x+2)b(x+2)}{b(x+1)} \;<\; \frac{(x+1)b(x+1)}{b(x)}.$$

It follows from (4) that the left hand side equals $S(\mathbb{N}) - \sum_{L\in\mathcal{J}(x+1)} \bar{W}(L)S(L)$ while the right hand side equals $S(\mathbb{N}) - \sum_{J\in\mathcal{J}(x)} \bar{W}(J)S(J)$. Thus the assertion is equivalent to

$$\sum_{J\in\mathcal{J}(x),L\in\mathcal{J}(x+1)} W(J)W(L)\big(S(L)-S(J)\big) \ > \ 0. \tag{8}$$

But each pair $(J,L) \in \mathcal{J}(x)\times\mathcal{J}(x+1)$ is uniquely determined by the three sets $M := J\cap L$, $K := (J\setminus M)\cup(L\setminus M)$ and $L' := L\setminus M$, and

$$W(J)W(L) \ = \ W(M)^2W(K) \quad \text{and} \quad S(L)-S(J) \ = \ 2S(L')-S(K).$$

Moreover, $\#K = 2x+1-2\#M$ and $\#L' = x+1-\#M$. Hence, the left hand side of (8) equals

$$\sum_{s=0}^{x} \sum_{M\in\mathcal{J}(s)} \sum_{K\in\mathcal{J}(2x+1-2s)} 1_{[M\cap K=\emptyset]} W(M)^2W(K)H(K) \tag{9}$$

with

$$\begin{aligned} H(K) \ &:= \sum_{L'\subset K\,:\,\#L'=x+1-s} \big(2S(L')-S(K)\big) \\ &= \sum_{i\in K} q_i \sum_{L'\subset K\,:\,\#L'=x+1-s} (2\cdot 1_{L'}(i)-1) \\ &= S(K)\binom{2x-2s}{x-s}\Big/(x+1-s). \end{aligned}$$

Hence, all summands in (9) are non-negative, and $W(M)^2W(K)S(K) > 0$ for suitable sets $M\in\mathcal{J}(x)$ and $K\in\mathcal{J}(1)$ with $M\cap K=\emptyset$. $\Box$

2.3 Log-density ratios along a ray

In what follows we consider the sequence $t\boldsymbol{p}$ for arbitrary $t\in(0,1]$, leading to the distributions $Q_{t\boldsymbol{p}}$ with probability mass functions $b_{t\boldsymbol{p}}$, weights $W_{t\boldsymbol{p}}(J)$ and sums $S_{t\boldsymbol{p}}(J)$. The corresponding Poisson probability mass functions are $\pi_{t\lambda}$, and this leads to the ratios $r_{t\boldsymbol{p}}$. According to Proposition 1,

$$f(t) \ := \ \log\rho(Q_{t\boldsymbol{p}}, \mathrm{Poiss}_{t\lambda}) \ = \ \max_{1\le x\le\lceil t\lambda\rceil} \log r_{t\boldsymbol{p}}(x) \ = \ \max_{1\le x\le\lceil\lambda\rceil} \log r_{t\boldsymbol{p}}(x).$$

Now we analyze the functions $L_x : (0,1]\to\mathbb{R}$,

$$\begin{aligned} L_x(t) \ &:= \ \log r_{t\boldsymbol{p}}(x) \\ &= t\lambda + \log\Big((t\lambda)^{-x}x! \sum_{J\in\mathcal{J}(x)} \prod_{i\in J} \frac{tp_i}{1-tp_i} \prod_{k\ge1}(1-tp_k)\Big) \\ &= t\lambda + \sum_{k\ge1}\log(1-tp_k) + \log\Big(\lambda^{-x}x! \sum_{J\in\mathcal{J}(x)} \prod_{i\in J} \frac{p_i}{1-tp_i}\Big), \end{aligned}$$

for integers $x \geq 0$ with $b(x) > 0$. Note first that $L_x(t)$ can be extended to a real-analytic function of $t \in (-\infty, 1/p_*) \supset [0,1]$, and

$$\begin{aligned} L_x(0) &= \log\Bigl(\lambda^{-x} x! \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} p_i\Bigr) \\ &\leq \log\Bigl(\lambda^{-x} \sum_{i(1),\ldots,i(x) \geq 1} \prod_{s=1}^{x} p_{i(s)}\Bigr) \;=\; \log(\lambda^{-x}\lambda^x) \;=\; 0 \end{aligned}$$

with equality for $x = 0, 1$ and strict inequality for $x > 1$. This shows already that f is a Lipschitz-continuous function on $(0,1]$ with limit $f(0+) = 0$.

Concerning the first derivative of L_x, for $t \in (0,1]$,

$$\frac{d}{dt} \prod_{i \in J} \frac{p_i}{1 - tp_i} \;=\; \sum_{k \in J} \frac{p_k^2}{(1 - tp_k)^2} \prod_{i \in J \setminus \{k\}} \frac{p_i}{1 - tp_i} \;=\; \prod_{i \in J} \frac{p_i}{1 - tp_i} \sum_{k \in J} \frac{p_k}{1 - tp_k},$$

whence

$$\begin{aligned} L_x'(t) &= \lambda - \sum_{k \geq 0} \frac{p_k}{1 - tp_k} + \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{p_i}{1 - tp_i} \sum_{k \in J} \frac{p_k}{1 - tp_k} \Big/ \sum_{J \in \mathcal{J}(x)} \prod_{i \in J} \frac{p_i}{1 - tp_i} \\ &= \lambda - \frac{1}{t}\Bigl(S_{t\boldsymbol{p}}(\mathbb{N}) - \sum_{J \in \mathcal{J}(x)} \bar{W}_{t\boldsymbol{p}}(J) S_{t\boldsymbol{p}}(J)\Bigr) \\ &= \lambda - \frac{1}{t} \sum_{J \in \mathcal{J}(x)} \bar{W}_{t\boldsymbol{p}}(J) S_{t\boldsymbol{p}}(J^c). \end{aligned}$$

Combining this formula with (4) yields

$$\begin{aligned} L_x'(t) &= \lambda - \frac{1}{t} \frac{(x+1) b_{t\boldsymbol{p}}(x+1)}{b_{t\boldsymbol{p}}(x)} \\ &= \lambda - \lambda \frac{r_{t\boldsymbol{p}}(x+1)}{r_{t\boldsymbol{p}}(x)} \\ &= \lambda\bigl(1 - \exp\bigl(L_{x+1}(t) - L_x(t)\bigr)\bigr). \end{aligned} \tag{10}$$

In particular,

$$L_x'(t) \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} 0 \quad \text{if and only if} \quad L_x(t) \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} L_{x+1}(t). \tag{11}$$

There is also an explicit expression for the second derivative of L_x: If $b(x+1) = 0$, then $x = n = \#\{i \geq 1 : p_i > 0\}$ and $L_x(t) = \lambda t + \log(\lambda^{-n} n!\, b(n))$, whence $L_x'' \equiv 0$. Otherwise, for $0 < t \leq 1$,

$$L_x''(t) \;=\; \lambda \exp\bigl(L_{x+1}(t) - L_x(t)\bigr)\bigl(L_x'(t) - L_{x+1}'(t)\bigr),$$

and

$$L_x'(t) - L_{x+1}'(t) \;=\; \frac{1}{t}\Bigl(\frac{(x+2) b_{t\boldsymbol{p}}(x+2)}{b_{t\boldsymbol{p}}(x+1)} - \frac{(x+1) b_{t\boldsymbol{p}}(x+1)}{b_{t\boldsymbol{p}}(x)}\Bigr) \;<\; 0$$

by Proposition 2. Hence L_x defines a smooth concave function on $[0,1]$.

3 Proofs of the main results

Proof of Theorem 1. We know that $f(t) = \log \rho(Q_{t\boldsymbol{p}}, \mathrm{Poiss}_{t\lambda})$ is equal to the maximum of $L_x(t)$ over $x \in \{1, \ldots, \lceil\lambda\rceil\}$, and that $f(0+) = 0$. Note also that

$$f'(t+) \;=\; \max_{x \in N(t)} L'_x(t)$$

where

$$N(t) \;:=\; \operatorname*{arg\,max}_{x \in \{1,\ldots,\lceil\lambda\rceil\}} \; r_{t\boldsymbol{p}}(x).$$

Since $g(t) := -\log(1 - tp_*)$ satisfies $g(0) = 0$ and $g'(t) = p_*/(1 - tp_*)$, it suffices to show that

$$L'_x(t) \;\le\; \frac{p_*}{1 - tp_*} \quad \text{for any } x \in N(t).$$

According to (10), the latter requirement is equivalent to

$$\frac{(x+1)b_{t\boldsymbol{p}}(x+1)}{b_{t\boldsymbol{p}}(x)} \;\ge\; t\lambda - \frac{tp_*}{1 - tp_*} \quad \text{for any } x \in N(t).$$

Note that $x \in N(t)$ implies that $L_{x-1}(t) \le L_x(t)$. But the latter inequality is equivalent to $L'_{x-1}(t) \le 0$, see (11), and by (10), this is equivalent to

$$\frac{xb_{t\boldsymbol{p}}(x)}{b_{t\boldsymbol{p}}(x-1)} \;\ge\; t\lambda.$$

Consequently, it suffices to show that

$$\frac{(x+1)b_{t\boldsymbol{p}}(x+1)}{b_{t\boldsymbol{p}}(x)} \;\ge\; t\lambda - \frac{tp_*}{1 - tp_*} \quad \text{whenever} \quad \frac{xb_{t\boldsymbol{p}}(x)}{b_{t\boldsymbol{p}}(x-1)} \;\ge\; t\lambda.$$

We may simplify notation by replacing $t\boldsymbol{p}$ with $\boldsymbol{p}$ and prove that

$$\frac{(x+1)b(x+1)}{b(x)} \;\ge\; \lambda - \frac{p_*}{1 - p_*} \quad \text{whenever} \quad \frac{xb(x)}{b(x-1)} \;\ge\; \lambda. \tag{12}$$

Note that for $1 \le x \le \lceil\lambda\rceil$, the representation (5) with $x - 1$ in place of x reads

$$\frac{b(x-1)}{xb(x)} \;=\; \sum_{J \in \mathcal{J}(x)} \bar{W}(J) \frac{1}{x} \sum_{i \in J} \frac{1}{q_i + S(J^c)}.$$

By Jensen's inequality,

$$\frac{1}{x} \sum_{i \in J} \frac{1}{q_i + S(J^c)} \;\ge\; \Bigl(\frac{1}{x} \sum_{i \in J} (q_i + S(J^c))\Bigr)^{-1} \;=\; \bigl(\bar{S}(J) + S(J^c)\bigr)^{-1},$$

so

$$\frac{b(x-1)}{xb(x)} \;\ge\; \sum_{J \in \mathcal{J}(x)} \bar{W}(J) \bigl(\bar{S}(J) + S(J^c)\bigr)^{-1}.$$

A second application of Jensen's inequality yields that

$$\frac{b(x-1)}{xb(x)} \;\geq\; \Big(\sum_{J \in \mathcal{J}(x)} \bar{W}(J)\big(\bar{S}(J) + S(J^c)\big)\Big)^{-1}.$$

Consequently, if $xb(x)/b(x-1) \geq \lambda$, then

$$\sum_{J \in \mathcal{J}(x)} \bar{W}(J)\big(\bar{S}(J) + S(J^c)\big) \;\geq\; \lambda.$$

On the other hand, (4) yields

$$\begin{aligned} \frac{(x+1)b(x+1)}{b(x)} \;&=\; \sum_{J \in \mathcal{J}(x)} \bar{W}(J)\big(\bar{S}(J) + S(J^c)\big) - \sum_{J \in \mathcal{J}(x)} \bar{W}(J)\bar{S}(J) \\ &\geq\; \lambda - \frac{p_*}{1-p_*}, \end{aligned}$$

because $\bar{S}(J) = x^{-1}\sum_{i \in J} p_i/(1-p_i) \leq p_*/(1-p_*)$ for any set $J \in \mathcal{J}(x)$. This proves (12). $\square$

Proof of Theorem 2. We know from Proposition 1 that in case of $\lambda \leq 1$,

$$\log \rho(Q, \mathrm{Poiss}_\lambda) \;=\; \log r(1) \;=\; L_1(1)$$

with

$$L_1(t) \;=\; t\lambda + \sum_{i \geq 1} \log(1 - tp_i) + \log\Big(\lambda^{-1} \sum_{i \geq 1} \frac{p_i}{1 - tp_i}\Big).$$

First of all, $L_1(0) = 0$, and

$$\begin{aligned} L_1'(t) \;&=\; \lambda - \sum_{i \geq 1} \frac{p_i}{1 - tp_i} + \sum_{i \geq 1} \frac{p_i^2}{(1 - tp_i)^2} \Big/ \sum_{i \geq 1} \frac{p_i}{1 - tp_i} \\ &=\; -t \sum_{i \geq 1} \frac{p_i^2}{1 - tp_i} + \sum_{i \geq 1} \frac{p_i^2}{(1 - tp_i)^2} \Big/ \sum_{i \geq 1} \frac{p_i}{1 - tp_i}, \end{aligned}$$

whence $L_1'(0) = \Delta$. Moreover, we have seen before that $L_1'' \leq 0$ by ultra-log-concavity of the probability mass functions $b_{t\boldsymbol{p}}$. Consequently, for some $\xi \in (0, 1)$,

$$L_1(1) \;=\; L_1(0) + L_1'(0) + 2^{-1}L_1''(\xi) \;=\; 0 + \Delta + 2^{-1}L_1''(\xi) \;\leq\; \Delta.$$

As to the lower bound, recall that

$$L_1(1) \;=\; \sum_{i \geq 1} (p_i + \log(1 - p_i)) + \log\Big(\lambda^{-1} \sum_{i \geq 1} \frac{p_i}{1 - p_i}\Big).$$

On the one hand,

$$p_i + \log(1 - p_i) \;=\; -\sum_{k \geq 2} \frac{p_i^k}{k} \;\geq\; -\frac{p_i^2}{2} \sum_{\ell \geq 0} p_*^\ell \;=\; -\frac{p_i^2}{2(1 - p_*)},$$

so

$$\sum_{i\ge 1}(p_i + \log(1-p_i)) \ \ge\ -\frac{1}{2(1-p_*)}\sum_{i\ge 1} p_i^2 \ =\ -\frac{\lambda}{2(1-p_*)}\Delta.$$

Moreover,

$$\log\Bigl(\lambda^{-1}\sum_{i\ge 1}\frac{p_i}{1-p_i}\Bigr) \ \ge\ \log\Bigl(\lambda^{-1}\sum_{i\ge 1}(p_i + p_i^2)\Bigr) \ =\ \log(1+\Delta) \ \ge\ \Delta - \Delta^2/2,$$

and this implies the asserted lower bound for $L_1(1)$. $\square$

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